Political Science 352

Fall 2002

Handout 2

Let S (and sometimes X) be sets of outcomes with typical elements x, y, z, or, sometimes, $x_1, x_2, \ldots, x_n$.

Preference relations

Def$\underline{^{\mathbf{n}}}$: $x \succ_i y$ means that person i *strictly prefers* outcome x to outcome y.

Def$\underline{^{\mathbf{n}}}$: : $x \succeq_i y$ ("x is weakly preferred to y by i") if it is not the case that $y \succ_i x$.

Or equivalently, not $y \succ x \Leftrightarrow x \succeq y$. The $\Leftrightarrow$ means or reads "if and only if."

Def$\underline{^{\mathbf{n}}}$: : $x \sim_i y$ (i is "indifferent" between outcomes x and y) if it is not the case that $x \succ_i y$ and it is not the case that $y \succ_i x$.

An implication: not $x \succ y \Leftrightarrow y \succeq x$, and not $y \succ x \Leftrightarrow x \succeq y$. So we could also write $x \sim_i y$ if $x \succeq y$ and $y \succeq x$.

Axioms about preferences

A1: (completeness) For any $x, y \in S$, either $x \succeq y$ or $y \succeq x$.

A2: (Transitivity of $\succeq$) For all $x, y, z \in S$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

Numerical representations of preferences (ordinal utility)

Defn: $u : S \to R$ is a numerical representation of $\succeq$ on S if

$$u(x) \geq u(y) \Leftrightarrow x \succeq y.$$

Equivalent ordinal utility functions

Claim: If $u(x)$ represents a person's preferences $\succeq$ on S in the sense just defined, then so does any function $v(x)$ that is a monotonic (i.e., order preserving) transformation of $u(x)$.

Conditions for an ordinal utility representation to exist

Thm: For S finite, there exists a numerical representation u of $\succeq$ iff $\succeq$ satisfies A1 and A2.

Sketch of Proof:

Two parts: (1) If there exists a u that is a n.r. then A1 and A2 must be satisfied.

Since $u(x)$ and $u(y)$ are numbers, it is the case that either $u(x) \geq u(y)$ or $u(y) \geq u(x)$. So, from the defn of a numerical representation, $u(x) \geq u(y)$ implies $x \succeq y$ and $u(y) \geq u(x)$ implies $u(y) \geq u(x)$, so completeness of $\succeq$ must obtain.

Similarly for transitivity.

Second part: If a dmkr satisfies A1 and A2, then there exists a u that is a numerical representation of $\succeq$.

To prove this, show *by construction*, an important proof technique. Use the axioms to show how to construct a u that works.

First step: Show that A1 and A2 imply that we can order the outcomes in S as $x_i \succeq x_j \succeq x_k \ldots$, using $\sim$ when appropriate. Completeness implies that dmkr can say "$x \succeq y$" or "$y \succeq x$" to every pairing, and if it were not possible to order the outcomes like this then transitivity (A2) must be violated for some triple of outcomes.

Second step: Assign numbers to each outcome in descending order, assigning the same number whenever $x_i \sim x_j$.

Third step: Show that by construction this u fn is a numerical representation of dmkr's preferences in the sense defined above.

Axioms, etc., using the strict preference relation

A1$'$: (asymmetry) For all $x, y \in S$, $x \succ y$ implies not $y \succ x$.

Claim: A1 is true iff A1$'$ if true.

Proof: (will use the technique of proof by contradiction, in which one assumes the claim is false and then shows that this generates a contradiction).

First, suppose A1 is true and A1$'$ is false. This would mean that

$$\text{A1 true: for all } x, y \in S, \text{ either } x \succeq y \text{ or } y \succeq x, \text{ AND}$$
$$\text{A1}' \text{ false: for some } x, y \in S, \ x \succ y \text{ and } y \succ x.$$

Remember that $\succeq$ and $\succ$ are defined in terms of each other. Using the fact that $x \succeq y$ implies not $y \succ x$ by definition, we can rewrite this

$$\text{A1 true: for all } x, y \in S, \text{ either not } y \succ x \text{ or not } x \succ y, \text{ AND}$$
$$\text{A1}' \text{ false: for some } x, y \in S, \ x \succ y \text{ and } y \succ x.$$

But notice now that A1 and A1$'$ cannot both be true. So we have proven that if A1 is true then so must A1$'$ be also.

What about the other direction? Suppose next that A1$'$ is true and A1 is false. Try to do it.

A2$''$: (Transitivity of $\succ$) For all $x, y, z \in S$ such that $x \succ y$ and $y \succ z$, it is true that $x \succ z$.

A2$'$: (negative transitivity) For any $x, y, z \in S$, if $x \succ y$, then either $x \succ z$ or $z \succ y$ or both.

Claim: A2 is true if and only if A2$'$ is true.

Claim: A2$'$ implies A2$''$, but A2$''$ does not imply A2$'$.

Lotteries, or gambles

Def$^{\underline{\text{n}}}$: Let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of outcomes.

$$\Delta(X) = \{(p_1, p_2, \ldots, p_n) : \sum_{i=1}^{n} p_i = 1 \text{ and } p_i \geq 0 \text{ for all } i\}$$

is the set of all probability distributions on the set X, which we will also call the set of *lotteries* or *gambles* on X.

The "natural principle"

A lottery that puts more weight (in terms of probability) on *good* outcomes (from the point of view of the decision maker), and less weight on *bad* outcomes is preferable to a lottery that puts more weight on the bad outcomes and less on the good ones.

An "ordinal utility" representation of preferences over lotteries

If a person's preferences over lotteries in $\Delta(X)$ satisfy A1, A2, and A3 (continuity, not given yet) then there is a utility function $u : \Delta(X) \to \mathbb{R}$ that represents these preferences in the sense that for all $p, q \in \Delta(X)$, $p \succ q$ iff $u(p) > u(q)$.

To observe: Preferences that can be represented in this way need not satisfy the natural principle.

Expected value

Def$^{\underline{\text{n}}}$: The *expected value* of a lottery $p \in \Delta(X)$ is defined as

$$EV(p) = \sum_{i=1}^{n} p_i x_i.$$

Note that $EV(p)$ is defined only if the set of outcomes is a set of numbers.

An expected value representation of preferences over lotteries

Def$\underline{^{\mathbf{n}}}$: A person's preferences are represented by the expected value function iff for p, $q \in \Delta(X)$, $p \succ q \Leftrightarrow EV(p) > EV(q)$.

The expected utility representation of preferences on $\Delta(X)$

Def$\underline{^{\mathbf{n}}}$: A person's preferences over the lotteries in $\Delta(X)$ (where X is a set of outcomes) can be represented by an *expected utility function* if there exists a function $u : X \rightarrow \mathbb{R}$ such that for any $p, q \in \Delta(X)$,

$$p \succ q \Leftrightarrow EU(p) \equiv \sum_{i=1}^{n} p_i u(x_i) > \sum_{i=1}^{n} q_i u(x_i) \equiv EU(q).$$

Equivalent expected utility functions

Claim: If $u(x)$ is an expected utility function that represents a person's preferences over gambles in $\Delta(X)$, then so is any $v(x) = au(x) + b$, where $a > 0$ and $a, b \in \mathbb{R}$.

Proof: Suppose $u(x)$ represents $\succ$ on $\Delta(X)$ in the sense that for any $p, q \in \Delta(X)$, $p \succ q \Leftrightarrow EU(p) > EU(q)$. Thus,

$$\begin{aligned}
\sum p_i u(x_i) &> \sum q_i u(x_i) \text{ (recall what } p_i, q_i \text{ are)} \\
a \sum p_i u(x_i) &> a \sum q_i u(x_i) \\
a(\sum p_i u(x_i)) + b &> a(\sum q_i u(x_i)) + b \\
\sum p_i (au(x_i) + b) &> \sum q_i (au(x_i) + b) \\
\sum p_i v(x_i) &> \sum q_i v(x_i).
\end{aligned}$$