Political Science 352

Fall 2002

Handout 3

The expected utility representation

Def$^{\underline{n}}$: A person's preferences over the lotteries in $\Delta(X)$ (where X is a set of outcomes) can be represented by an *expected utility function* if there exists a function $u : X \to \mathbb{R}$ such that for any $p, q \in \Delta(X)$,

$$p \succ q \Leftrightarrow EU(p) \equiv \sum_{i=1}^{n} p_i u(x_i) > \sum_{i=1}^{n} q_i u(x_i) \equiv EU(q).$$

Axioms that imply EU-representable preferences

Consider a set of outcomes X, and lotteries $\Delta(X)$.

A1 (completeness): For any $p, q \in \Delta(X)$, if $p \succ q$ then not $q \succ p$.

A2 (neg. transitivity): For any $p, q, r \in \Delta(X)$, if $p \succ q$, then either $p \succ r$ or $r \succ q$. (Recall that neg trans implies trans.)

A3 ("Archimedian", or continuity): For $p, q, r \in \Delta(X)$ s.t. that $p \succ q \succ r, \exists \alpha, \beta \in (0,1)$ s.t. $(\alpha p, (1-\alpha)r) \succ q \succ (\beta p, (1-\beta)r)$.

A4 (substitution, or "independence"): For any $p, q, r \in \Delta(X)$, if $p \succ q$, then for $\alpha \in (0,1)$, $(\alpha p, (1-\alpha)r) \succ (\alpha q + (1-\alpha)r)$.

Th$^{\underline{m}}$ (: a) A dmkrs prefs $\succ$ on $\Delta(X)$ can be represented by an expected utility function iff they satisfy A1, A2, A3, and A4.

Compound lotteries

Objects like $(\alpha p, (1-\alpha)r)$ are *compound lotteries* – the lottery you get when you have an α chance of getting the lottery p, and a $1-\alpha$ chance of getting the lottery r.

The Allais paradox, an example of a common violation of A4

You have a choose between two gambles:

G1 = \$1m, G2 = (.01 on \$0, .89 on \$1m, .10 on \$5m), and then between two more,

G3 = (.89 on 0, .11 on 1m), G4 = (.9 on 0, .1 on 5m).

People often express the preferences $G1 \succ G2$ and $G4 \succ G3$. But this violates the substitution axiom, as show by

	probability			vs.		probability		
	.01	.10	.89			.01	.10	.89
G1	1m	1m	1m		G3	1m	1m	0m
G2	0m	5m	1m		G4	0m	5m	0m

Now we show that you can't represent these expressed preferences with an EU function (which we already know must be the case from the Theorem above):

Let $X = \{0, 1\text{m}, 5\text{m}\}$. Let $u(5m) = 1$, $u(0) = 0$, and keep $u(1m)$ "free." (Why can we do this without losing any generality?)

G1 $\succ$ G2 implies $u(1m) > .01 * 0 + .89 * u(1m) + .10 * 1$, but this implies

$$u(1m) > .89u(1m) + .1$$

or

$$.11u(1m) > .1$$

And G3 $\prec G4$ implies

$$.89u(0) + .11u(1m) < .9u(0) + .1u(5m)$$

$$.11u(1m) < .1$$

But this is a contradiction, so there is no eu function that can represent these preferences.

Utility functions over continuous sets of outcomes and risk attitudes

Def$^{\underline{n}}$: A person is *risk averse* if she strictly prefers to receive the expected value of a lottery for sure to the lottery itself. Formally, if for numerical outcomes x and y, she has

$$\alpha x + (1 - \alpha)y \succ (\alpha x, (1 - \alpha)y) \text{ for all } \alpha \in (0, 1).$$

Two concepts of risk attitude

1) Absolute risk attitude: The idea just presented, curvature of a utility function defined on a continuous, numerical set of outcomes.

2) Relative risk attitude: "Rumsfeld is more risk acceptant than Powell." Here, idea is that A is more risk acceptant (averse) than B if A will take (reject) gambles that B would reject (accept). This does not presume any underlying metric in outcomes.

Definition of a normal form game

A normal form game is

1. A set of players $I = \{1, 2, \ldots, n\}$.
2. A set of strategies available to each player i, S_i.
3. A utility function $u_i : S_1 \times S_2 \times \ldots \times S_n \to \mathbb{R}$ for each player i.

Formally, we will often write $\Gamma = \langle I, S_i, u_i \rangle$.

Payoff notation

In general, $u_1(s_i, s_j)$ is the utility number for player 1 assigned when 1 chooses strategy $s_i \in S_1$ and player 2 chooses $s_j \in S_2$.

What if the number of players is more than 2? Then $u_1(s_1, s_2, s_3, \ldots)$ is the utility payoff player 1 gets when she chooses s_1 and player 2 chooses $s_2 \in S_2$, etc. For example, $u_1(F, F, F, P, F, P)$ means what in a six player SoN game?

For convenience, we will often write (for $n > 2$ player games) $u_i(s_i, s_{-i})$ for player i's payoff, where s_i is the strategy i is choosing and s_{-i} is the list of strategies that everyone else is choosing. ($-i$ here means something like "not person i".) Thus in the example in the last paragraph, $s_i = F$ and $s_{-i} = (F, F, P, F, P)$.

Payoffs given beliefs about what other player may choose

We can extend the notation for utility payoffs for outcomes to represent a player's *expected* utility for choosing a given strategy s_i *given* that the player has some belief about what the other is likely to do.

Consider a two player game in which 1's strategies are $S = \{s_1, s_2, s_3, \ldots, s_m\}$ and 2's strategies are $T = \{t_1, t_2, \ldots, t_n\}$.

Then 1's belief about player 2's likely play is a probability distribution on T, thus an element $\tau \in \Delta(T)$. Written out, $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$, where $\tau_i = Pr(2 \text{ plays } t_i)$.

For the general case where 1 plays some strategy $s_i \in S$,

$$u_1(s_i, \tau) = \sum_{j=1}^{n} \tau_j u_1(s_i, t_j).$$

Definition of a best reply

Def$^{\underline{\mathbf{n}}}$: s_i is a *best reply given belief* τ for player 1 if she has $u_1(s_i, \tau) \geq u_1(s_j, \tau)$ for all strategies $s_j \in S_1$.

Def$^{\underline{\mathbf{n}}}$: The set of best replies for player 1 is $BR_1(\tau) = \{s_i : s_i \text{ is a best reply given } \tau\}$.

(Note that the ":" in the last definition reads "such that.")

Never a best reply

First concept of a *bad strategy* that a rational player could rule out:

Def$^{\underline{\mathbf{n}}}$: $s_i \in S$ is *never a best reply* if there does not exist $\tau \in \Delta(T)$ s.t. $u(s_i, \tau) \geq u(s, \tau)$ for all $s \in S$.

(Note: Now we are talking about two player games and I am using S for 1's strategy set and T for 2's strategy set.)

Question: Is it true in general that if a strategy s_i gives better payoffs than another strategy s_j for every possible strategy choice by player 2, then it is does better for any belief $\tau \in \Delta(T)$? Yes.

Claim: For two strategies (acts) s_i and s_j, $u(s_i, \tau) > u(s_j, \tau)$ for all $\tau \in \Delta(T)$ iff $u(s_i, t) > u(s_j, t)$ for all $t \in T$.

The idea of mixed strategies

Suppose we give players the option of choosing a probability distribution on the set of available pure strategies.

If 1's set of pure strategies (actions, acts) is $S = \{s_1, s_2, \ldots, s_m\}$, then $\Delta(S)$ is the set of mixed strategies.

This is exactly parallel to the idea of $\tau = (\tau_1, \tau_2, \ldots)$, except there we were talking about τ as player 1's beliefs about 2's likely play, and here we are talking about σ as 1's actual strategy.

If player 1 has m available strategies in an $m \times n$ normal form game, we write

$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$, where σ_i is the probability that player 1 choose strategy $s_i \in S$.

Expected payoff given a mixed strategy σ and a belief τ

Defn: A person's eu for the mixed strategy $\sigma \in \Delta(S)$, given beliefs about the other's play $\tau \in \Delta(T)$, is

$$u(\sigma, \tau) = \sum_{i=1}^{m} \sigma_i u(s_i, \tau) = \sum_{i=1}^{m} \sigma_i \sum_{j=1}^{n} \tau_j u(s_i, t_j).$$

Strongly dominated strategies

A second concept of what a "bad strategy" would be for a rational player in a game situation:

Defn: $\sigma \in \Delta(S)$ is *strongly dominated* by $\sigma' \in \Delta(S)$ if $u(\sigma', t) > u(\sigma, t)$ for all $t \in T$.

In words, strategy σ gives a worse expected payoff for player 1 tahn does strategy σ' no matter what player 2 chooses.

Equivalence of the two concepts in 2 person games

Th$^{\underline{\mathbf{m}}}$ :: If there are 2 players, then $\sigma \in \Delta(S)$ is never a best reply iff there exists $\sigma' \in \Delta(S)$ s.t. σ is strongly dominated by σ'.

In words, if there is no set of beliefs for which a strategy is optimal (a best reply), then there must exist some other strategy that is better than it in all possible circumstances regarding the other player's play.