Solutions to Problem Set 3
Political Science 152/352

Question 1

(a) Set of rationalizable pure strategy profiles is the whole strategy profile space, i.e. it is $\{U, M, D\} \times \{L, C, R\}$. One way to see this is to note that nothing in this game can be eliminated by strict domination and to remember that in two person games the set of rationalizable strategies is the same as the set of strategies that survive iterated elimination of strictly dominated strategies. One can also check directly that each strategy is rationalizable. Now, U is rationalized by L. For this to make U rationalizable we need L to be rationalizable also. L is rationalized by M. But again, to conclude that L is rationalizable we need M to be rationalizable. M is rationalized by C and C is rationalized by U; which closes the loop; i.e. with these arguments we have shown that U, M for player 1 and L, C for player 2 are rationalizable. Also, D and R rationalize each other. Hence everything is rationalizable.

(b)It is easy to check that only pure strategy Nash Equilibrium (NE) is (D,R).

There are no mixed strategy NE of this game. The strategy we will follow to show this is to first rule out the possibility of a mixed strategy NE in which player 1 plays all her strategies with positive probability, then rule out possibility of her playing any two of her strategies with positive probability. This will be sufficient by either noting that the game is symmetric or noting that player 2 has a unique pure strategy best response to every pure strategy of player 1.

Assume player 2 plays L with probability τ_1, C with probability τ_2 and R with probability $1 - \tau_1 - \tau_2$.

Step 1 - rule out the possibility that player 1 plays all her strategies with positive probability in a NE
If player 1 plays all her strategies with positive probability it must be the case that she is indifferent between all of them. For her to be indifferent between U and D, we need $\tau_1 = \tau_2 \leq \frac{1}{2}$. For R to give the same expected utility we need $10\tau_1 - (1 - 2\tau_1) = 12\tau_1 + 2(1 - 2\tau_1)$, i.e. $\tau_1 = \frac{3}{4} > \frac{1}{2}$. Hence, this is not possible.

Step 2 - rule out the possibility that player 1 plays U and M with positive probability and D with zero probability in a NE
For this to happen we again need $\tau_1 = \tau_2$. But whenever this is the case player 1 strictly prefers D to both U and M. This is because, in this case playing U or M will give player 1 a compound lottery that is equivalent to a lottery giving utility of 5 or -1, whereas playing D will give him a lottery between 6 and 2. And in each case the probability of good outcome is $\tau_1 + \tau_2$. Hence, this is not possible.

Step 3 - rule out the possibility that player 1 plays U and D with positive probability and M with zero probability in a NE
If player 1 plays M with zero probability then player 2 never plays L because it is strictly dominated by R. But if player 1 never plays L, player 2 will never play U. Hence, there is no such NE.

Step 4 - rule out the possibility that player 1 plays M and D with positive probability and U with zero probability in a NE
Similar to step 3, if player 1 does not play U, player 2 never plays C, in which case player 1 does not play M. Hence, this kind of equilibrium is also not possible.

Therefore, there are no mixed strategy NE of this game.

Question 2

(a) $S_i = X = \{0, 1, ..., 100\}$ for $i = 1, 2$. Note that a candidate wins if and only if the voter with ideal point 50 votes for him.
Therefore,

$$u_i(s_1, s_2) = \{ \begin{array}{l} 1 \text{ if } |s_i - 50| < |s_j - 50| \\ \frac{1}{2} \text{ if } |s_i - 50| = |s_j - 50| \\ 0 \text{ if } |s_i - 50| > |s_j - 50| \end{array}$$

(b)$s_i = 0$ or $s_i = 100$ are strictly dominated by $s_i = 50$. To see this note that for $|s_j - 50| = 50$, $s_i = 0$ and $s_i = 100$ gives a utility of $\frac{1}{2}$ to player i and $s_i = 50$ gives 1. For $s_j = 50$, $s_i = 0$ and $s_i = 100$ gives 0 and $s_i = 50$ gives $\frac{1}{2}$. For all other cases, $s_i = 0$ and $s_i = 100$ gives 0 and $s_i = 50$ gives 1.

(c) As noted above, 0 and 100 are strictly dominated by 50 and hence are eliminated for both players.
Consider $s_i = 1$. This strategy gives a utility of $\frac{1}{2}$ if $s_j = 1$ or if $s_j = 99$ and zero otherwise. On the other hand $s_i = 50$ gives utility of 1 if $s_j = 1$ or if $s_j = 99$ and still strictly positive utility otherwise. Therefore, at this step we can eliminate $s_i = 1$ and similarly $s_i = 99$ because they are strictly dominated by $s_i = 50$.
Note that this would not be the case if we had not eliminated $s_j = 0$ and $s_j = 100$ earlier, because $s_i = 1$ and $s_i = 50$ would both give a utility of 1 when $s_j = 0$, hence we would not get strict domination.
Proceeding similarly, it can be shown that the only strategy profile that survives iterated elimination of strictly dominated strategies is (50,50).

(d)$(50, 50)$ is a NE. To see this we need to check if any of the players has an incentive to deviate. If $s_1 = 50$, player 2 is getting $\frac{1}{2}$ by playing 50 and would get 0 with any other strategy. Hence player 2 - and similarly player 1 - has no incentive to deviate.

For uniqueness, suppose (s'_1, s'_2) is a NE. If $s'_1 < 50$ and $s'_2 > 50$, and if there is no tie the losing player will have an incentive to deviate to 50. If there is a tie both will have an incentive to deviate to 50. If $s'_1 < 50$ and $s'_2 < 50$ or if $s'_1 > 50$ and $s'_2 > 50$ then similarly, at least one of the players has an incentive to deviate to 50 (both if there is a tie). Therefore in any NE we need at least one player playing 50. But, 50 is the unique best response to 50. Hence, the unique NE of this game is (50,50).
An easier way to see that this equilibrium is unique is to note that the set of strategies that do not survive iterated elimination of strictly dominated strategies are never a best response. Since the strategies played in a NE would have to be a best response to at least to the other player's strategy in NE, they would not be eliminated by iterated elimination of strictly dominated strategies; i.e. all NE strategies survive iterated elimination of strictly dominated strategies. But in part (c) we showed that the only strategy profile that survives iterated elimination of strictly dominated strategies is (50,50) which is later shown to be a NE. Therefore it has to be the unique NE.

(e)It is tempting to argue that the convergence of both candidates to identical policy positions at "the middle of the road" is a bad thing, because it means that the voters have no real "choice" and people with different preferences (more towards the extremes) are effectively not represented. But on the other hand, the presumption is that the candidates are choosing and implementing public policies, which is to say that by their nature the policies have to apply to all the people in the jurisdiction – you can't make one public policy for some people and a different one for others unless you put them in different jurisdictions. If this is so, then arguably convergence at the median is a good thing, since if the distribution of voter preferences is symmetric, this is the policy position that minimizes the "average dissatisfaction" of all voters.

Question 3

(a) This is a 2×2 normal form game, which we can write as:

		Army B	
		Defend 1	Defend 2
Army A	Attack 1	$0, \ 0$	$v_1, -v_1$
	Attack 2	$1, -1$	$0, \ 0$

Notice that this game does not have a Nash equilibrium in pure strategies since one army will always have an incentive to deviate: Army A's best response to B's strategy is to attack the undefended target, while Army B's best response is to defend the target that A plan's to attack. Since this is a finite game, by

Nash's Theorem, a Nash equilibrium must exist, and so it must be in mixed strategies.

Let $p = \text{Prob(D1)}$ and let $q = \text{Prob(A1)}$. For Army B to induce Army A to play a mixed strategy, we must have $E[u_A(A1)] = E[u_A(A2)]$. Solving for p:

$$\begin{aligned} p \cdot 0 + (1-p) \cdot v_1 &= p \cdot 1 + (1-p) \cdot 0 \\ (1-p)v_1 &= p \\ v_1 &= p + pv_1 \\ p &= \frac{v_1}{v_1 + 1} \end{aligned}$$

Similarly, Army A's mixed strategy must make Army B indifferent between which target they attack, $E[u_B(D1)] = E[u_B(D2)]$. Next, solve for q:

$$\begin{aligned} q \cdot 0 + (1-q) \cdot (-1) &= q \cdot (-v_1) + (1-q) \cdot 0 \\ q - 1 &= -qv_1 \\ q + qv_1 &= 1 \\ q &= \frac{1}{v_1 + 1} \end{aligned}$$

Thus, there is a unique Nash equilibrium of this game where Army A plays the mixed strategy $(\frac{1}{v_1+1}\text{ A1 }, \frac{v_1}{v_1+1}A2)$ and Army B plays $(\frac{v_1}{v_1+1}\text{ D1 }, \frac{1}{v_1+1}D2)$.

(b) The probability that target 1 will be destroyed is

$$\begin{aligned} P_1 &= q(1-p) \\ &= \frac{1}{v_1+1}\left(1 - \frac{v_1}{v_1+1}\right) \\ &= \frac{1}{(v_1+1)^2} \end{aligned}$$

The probability that target 2 will be destroyed is:

$$\begin{aligned} P_2 &= (1-q)p \\ &= \left(1 - \frac{1}{v_1+1}\right)\frac{v_1}{v_1+1} \\ &= \frac{v_1^2}{(v_1+1)^2} \end{aligned}$$

As $v_1 \to \infty$, $P_1 \to 0$ and $P_2 \to 1$. This result is fairly intuitive. As the value of target 1 increases, Army B will be more likely to defend it, and as a consequence, Army A will be less and less likely to attack it.

Question 4

(a) To specify this as a normal form game, the set of players is $I = \{1, 2, \ldots, 100000\}$, the set of strategies is $S_i = \{P, H\}, \forall i$ (P for "Protest" and H for "stay at Home"). Let the integer n be the number of other people (not including player i) who decide to protest–this is a shorthand way of characterizing other players' strategies s_{-i}. The utility function for each player is

$$u_i(s_i, s_{-i}) = \begin{cases} -k-1 & \text{if } s_i = P, n \leq 599 \\ -\frac{600}{n+1}k - 1 & \text{if } s_i = P, n \in [600, 9,998] \\ 9 & \text{if } s_i = P, n \geq 9,999 \\ 0 & \text{if } s_i = H, n \leq 9,999 \\ 10 & \text{if } s_i = H, n \geq 10,000 \end{cases}$$

To determine player i's best response to s_{-i}, there are 4 cases. (For ease of notation, just let $u(\cdot)$ be the utility function above.) First, if $n \leq 599$, $u(P) = -k-1 < u(H) = 0 \Rightarrow BR = H$. Second if $n \in [600, 9,998], u(P) = -\frac{600}{n+1}k-1 < u(H) = 0 \Rightarrow BR = H$. Third, if $n = 9,999, u(P) = 9 > u(H) = 0 \Rightarrow BR = P$.

If $n \geq 10,000, u(P) = 9 < u(H) = 10 \Rightarrow BR = H$. To summarize, the best response function (in terms of n) for any player i is

$$BR(n) = \begin{cases} H & \text{if } n \leq 9,998 \text{ or } n \geq 10,000 \\ P & \text{if } n = 9,999 \end{cases}$$

This game has two types of pure strategy Nash equilibria. There is one equilibrium where everyone stays home, e.g. the strategy profile $S = (H, H, \ldots, H)$. To see that this is a Nash equilibrium, note that H is a best response to everyone else staying home, $n = 0$. The other set of equilibria occur where *exactly* 10,000 choose to protest since when $n = 9,999$, player i is "pivotal" in the sense that his action (given that n is fixed) decides whether the government collapses or not. This is clearly a severe type of coordination problem because any other number of protesters turning out is not a Nash equilibrium.

(b) For this normal form, I and S_i are the same as in part (a). The utility function for any given player i is slightly different:

$$u_i(s_i, s_{-i}) = \begin{cases} -k+1 & \text{if } s_i = P, n \leq 600 \\ -\frac{600}{n+1}k + 1 & \text{if } s_i = P, n \in [601, 9,998] \\ 11 & \text{if } s_i = P, n \geq 9,999 \\ 0 & \text{if } s_i = H, n \leq 9,999 \\ 10 & \text{if } s_i = H, n \geq 10,000 \end{cases}$$

In this case, the best response function depends on the value of k. First, suppose $k < \frac{9,999}{600}$. First, if $n \leq 599$, $u(P) = -k+1 < 0 = u(H)$, so $BR = H$ as in part (a). If $n \in [600, 600k-1]$, staying home is still a best response, since when $n = 600k-1, u(P) = -\frac{600k}{600k-1+1}+1 = u(H) = 0$. For $n \in [600k, 9998], n >$

$600k \Rightarrow \frac{600k}{n+1} < 1 \Rightarrow -\frac{600k}{n+1} + 1 > 0 \Rightarrow u(P) > u(H)$. Even though there are not enough people to force the overthrow of the government, the best response is to protest because the probability of being arrested is sufficiently small. For $n = 9999$, $u(P) = 11 > 0 = u(H)$ and for $n \geq 10000, u(P) = 11 > 10 = u(H)$, so the best response is to protest even if you are not pivotal in the sense of part (a).

Thus, if $n \leq 600k - 1$, the best response is H, and if $n \geq 600k - 1$, the best response is P. This implies there are exactly two Nash equilibria: one where everyone stays home, and one where everyone protests. The logic behind the first equilibrium is the same as in part (a). Given that a citizen expects everyone else to stay home, her best response is also to stay home. For the second equilibrium, if a citizen expects everyone to turn out and protest, then protesting is a best response.

There are no other Nash equilibria. If the strategy profile specifies that $600k$ or less people protest, then citizens who protest have an incentive to defect and stay home. If the strategy profile specifies that $600k$ or more (but less than $100,000$) people protest, then citizens who stay at home have an incentive to turn out and protest, too.

(c) The anniversaries of other protests are "focal points" and help to coordinate people's actions. In the model here, bringing down an authoritarian regime is a coordination problem. Given that one expects many others to turn out and protest on an anniversary, then it is also in one's interest to turn out on that day to protest.

Question 5

(a) Here, the normal form consists of $I = \{1,2\}, S_i \in [0,\infty)$ for $i = 1,2$, and utility functions:

$$\begin{aligned} u_1(s_1,s_2) &= p(s_1,s_2)v - k_1 s_1 \\ &= \frac{s_1}{s_1+s_2}v - k_1 s_1 \\ u_2(s_1,s_2) &= (1-p(s_1,s_2))v - k_2 s_2 \\ &= \frac{s_2}{s_1+s_2}v - k_2 s_2 \end{aligned}$$

To find the best reponse function, first differentiate u_i with respect to s_i for $i = 1,2$, set equal to zero and solve for s_i.

$$\begin{aligned} \frac{\partial u_1(s_1,s_2)}{\partial s_2} &= \frac{vs_2}{(s_1+s_2)^2} - k_1 = 0 \\ \frac{s_2 v}{k_1} &= (s_1+s_2)^2 \end{aligned}$$

$$\sqrt{\frac{s_2 v}{k_1}} = s_1 + s_2$$

$$s_1^* = \sqrt{\frac{s_2 v}{k_1}} - s_2$$

By symmetry,

$$s_2^* = \sqrt{\frac{s_1 v}{k_2}} - s_1$$

To solve for the Nash equilibrium, we solve these two equations for s_1 and s_2.

$$\begin{aligned}
s_1^* &= \sqrt{\frac{s_2^* v}{k_1}} - s_2 \\
s_1^* &= \sqrt{\frac{(\sqrt{\frac{s_1^* v}{k_2}} - s_1^*)v}{k_1}} - \sqrt{\frac{s_1^* v}{k_2}} - s_1^* \\
\sqrt{\frac{s_1^* v}{k_2}} &= \sqrt{\frac{(\sqrt{\frac{s_1^* v}{k_2}} - s_1^*)v}{k_1}} \\
\frac{s_1^* v}{k_2} &= \frac{(\sqrt{\frac{s_1^* v}{k_2}} - s_1^*)v}{k_1} \\
\frac{s_1^* k_1}{k_2} &= \sqrt{\frac{s_1^* v}{k_2}} - s_1^* \\
\frac{(k_1 + k_2)s_1^*}{k_2} &= \sqrt{\frac{s_1^* v}{k_2}} \\
\left(\frac{(k_1 + k_2)s_1^*}{k_2}\right)^2 &= \frac{s_1^* v}{k_2} \\
s_1^* &= \frac{v k_2}{(k_1 + k_2)^2}
\end{aligned}$$

And again, a symmetry argument gets us

$$s_2^* = \frac{v k_1}{(k_1 + k_2)^2}$$

(b) Let $k_2 = 1$, then

$$s_1 = \frac{v}{(k_1 + 1)^2}$$

$$s_2 = \frac{v k_1}{(k_1 + 1)^2}$$

As $k_1 \to 1, s_1 \to \frac{v}{4}, s_2 \to \frac{v}{4}$ and $p(s_1, s_2) \to \frac{1}{2}$. Intuitively, as the cost of 1 raising funds comes closer to 2's cost of raising funds, each candidate spends an equal amount and the probability of winning tends toward a 50-50 chance.

As $k_1 \to 0, s_1 \to v, s_2 \to 0$. As k_1 decreases, the incumbent becomes relatively more advantaged in terms of raising funds. When this happens, the equilibrium spending by the incumbent tends toward the value of office v, while the challenger's spending tends toward 0. The probability of the incumbent's re-election also goes to 1.