Solutions for Problem Set 4
Political Science 152/352

Question 1

Both tic-tac-toe and chess are finite dynamic games of complete and perfect information. (A chess game cannot go on forever because of the rule that the game is a draw if 50 consecutive moves are made without capturing a piece.) The appropriate solution concept is Subgame Perfect Nash Equilibrium. Since both tic-tac-toe and chess are finite games, we know that an equilibrium (solution) exists.

Question 2

(a) See Figure 1. There are two SGPNE of this game. Solving the game from the back, we find that Player 2's best response when $x \in \{0, 1, 2\}$ is to choose Y_x, and when $x = 3$ he is indifferent, so either Y_3 or N_3 would be optimal. One SGPNE is where $x = 3$ and $S_2 = Y_0Y_1Y_2Y_3$ and one where $x = 2$ and $S_2 = Y_0Y_1Y_2N_3$. The second equilibrium seems more plausible (where payoffs are $x = 2, 1 - x = 1$) since if $x = 3$, Player 2 will probably feel cheated and feel like there is no reason that Player 1 should get a payoff of 3. This story, however, implies that we have misspecified Player 2's payoffs.

(b) See Figure 2. Nash Equilibria are indicated with $*$.

(c) One Nash Equilibrium that is not subgame perfect occurs where $x = 0$ and $S_2 = Y_0N_1N_2N_3$. In words, Player 2 threatens to reject all offers where he does not receive the full amount of the pie, and since Player 1 is indifferent between all possible offers (he will always get payoff 0 given this strategy for Player 2). This is an example of a Nash Equilibrium that is ruled out by subgame perfection because the strategy for Player 2 involves non-credible (incredible?) commitments.

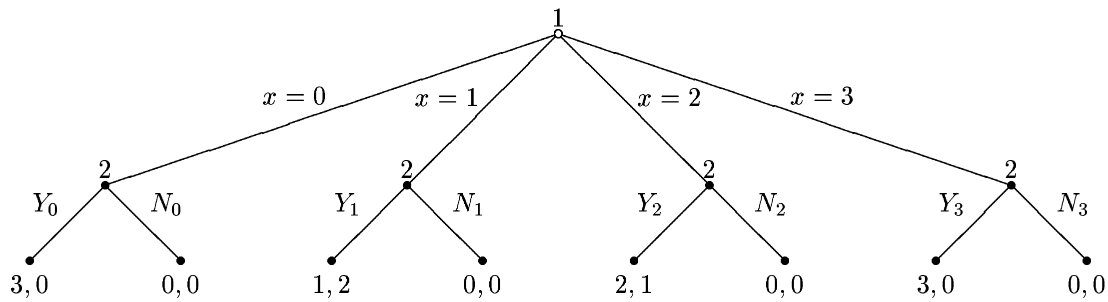

Figure 1: Extensive form for Question 2

		Player 1			
		x=0	x=1	x=2	x=3
Player 2	$Y_0Y_1Y_2Y_3$	$3,\ 0$	$2,1$	$1,2$	$0,3*$
	$Y_0Y_1Y_2N_3$	$3,0$	$2,1$	$1,2*$	$0,0$
	$Y_0Y_1N_2Y_3$	$3,0$	$2,1$	$0,0$	$0,3*$
	$Y_0Y_1N_2N_3$	$3,0$	$2,1*$	$0,0$	$0,0$
	$Y_0N_1Y_2Y_3$	$3,0$	$0,0$	$1,2$	$0,3*$
	$Y_0N_1Y_2N_3$	$3,0$	$0,0$	$1,2*$	$0,0$
	$Y_0N_1N_2Y_3$	$3,0$	$0,0$	$0,0$	$0,3*$
	$Y_0N_1N_2N_3$	$3,0*$	$0,0$	$0,0$	$0,0*$
	$N_0Y_1Y_2Y_3$	$0,0$	$2,1$	$1,2$	$0,3*$
	$N_0Y_1Y_2N_3$	$0,0$	$2,1$	$1,2*$	$0,0$
	$N_0Y_1N_2Y_3$	$0,0$	$2,1$	$0,0$	$0,3*$
	$N_0Y_1N_2N_3$	$0,0$	$2,1*$	$0,0$	$0,0$
	$N_0N_1Y_2Y_3$	$0,0$	$0,0$	$1,2$	$0,3*$
	$N_0N_1Y_2N_3$	$0,0$	$0,0$	$1,2*$	$0,0$
	$N_0N_1N_2Y_3$	$0,0$	$0,0$	$0,0$	$0,3*$
	$N_0N_1N_2N_3$	$0,0$	$0,0$	$0,0$	$0,0*$

Figure 2: Normal form for Question 2

Question 3

This problem is an application of take-it-or-leave-it bargaining to a situation of political agenda setting. It is commonly attributed to Tom Romer and Howard Rosenthal.

(a) The agency chooses a number b from the set $S_A = [0, \infty)$, so an example of a strategy for A is $b = 100$. A complete strategy for the median legislator must specify his action (either Accept or Reject) at each of his choice nodes; that is, for each possible value of b. Since each of the legislator's nodes corresponds to a different value of b, we can specify a complete strategy as a function of b, say $s_M : S_A \to \{Accept, Reject\}$. For example, if the legislator's strategy is to accept if the budget is no more than 10, then

$$s_M(b) = \begin{cases} Accept & \text{if } b \leq 10 \\ Reject & \text{if } b > 10 \end{cases}$$

(b) Using the suggestion, first consider the case where $b^* \geq b_m$. Since the game is a finite game of complete and perfect information, we can solve it using backwards induction. The legislator compares his utility from the proposed budget b and the reversion point b^* and chooses to accept if and only if $u_m(b) \geq u_m(b^*)$. Since the utility function is symmetric around the legislator's ideal point b_m, this implies that optimal play for M in each subgame proscribes *Accept* in all subgames when $b \in [\max(0, 2b_m - b^*), b^*]$ and *Reject* in all other subgames. Since A's utility is a linear monotonic function, the agency's best reply is to choose the highest b such that M will accept. Thus, $b = b^*$. Formally, the following strategies constitute a subgame perfect Nash equilibrium:

$$s_A = b^*$$

$$s_M(b) = \begin{cases} Accept & \text{if } b \in [\max(0, 2b_m - b^*), b^*] \\ Reject & \text{otherwise} \end{cases}$$

The intuition here is that although the agency would like to propose a higher budget, it cannot do any better than the reversion point because it anticipates that any proposals $b > b^*$ will be rejected by M. Note that any strategy profile where the agency proposes $b > b^*$ or $b \in [0, 2b_m - b^*]$ (when this interval is nonempty) and the legislator's strategy is the same as above will also be a subgame perfect Nash equilibrium. This is because on the equilibrium path, the legislator will reject such offers and the budget reverts to b^*.

In the case where $b^* < b_m$, optimal play in every subgame implies that M *Accepts* if $b \in [b^*, 2b_m - b^*]$ and *Rejects* for all other values of b. The highest b that the agency can propose is $b = 2b_m - b^*$, which is greater than the reversion point. This is the budget level that makes the legislator indifferent between the proposal b and the reversion point b^*. Formally, the SGPNE is:

$$s_A = 2b_m - b^*$$

$$s_M(b) = \begin{cases} Accept & \text{if } b \in [b^*, 2b_m - b^*] \\ Reject & \text{otherwise} \end{cases}$$

(c) See Figure 3. The agency's preference is to set as high a budget as possible, but it is constrained by the reversion point b^* and the legislator's preferences. For values of the reversion point such that $b^* \geq b_m$, the equilibrium outcome just tracks the reversion point $b^{eq} = b^*$ because the legislator will not accept any new budget that exceeds the reversion point, and so the agency is unable to move the budget further in its direction. This is the upward sloping part of the graph.

For values of $b^* < b_m$, the agency now has some structural bargaining power because it can move the budget in its favor while still leaving the legislator as well off as he was before. In fact, the lower b^* is, the greater the agency's power in terms of how far $(b^{eq} - b^*)$ it can move the budget from the reversion point. One implication of this model is that zero-based budgeting may be undesirable.

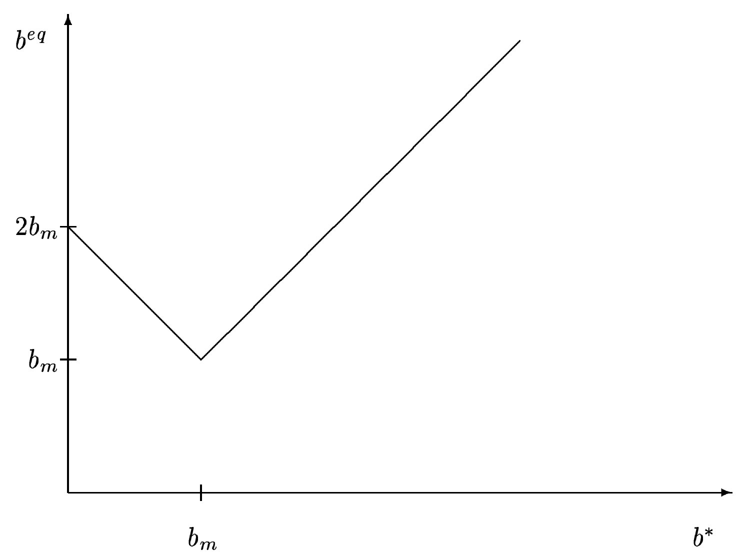

Figure 3: 3(c) Equilibrium budget outcomes

Question 4

(a) Below, in figure (a) the area enclosed by the diamond represents all possible per period stage game payoffs. The minmax payoff for each player is 0. Hence, according to the Folk theorem it is possible to support everything in the diamond, which is also in the positive ortant (that is, the area enclosed by the diamond in figure (b)) as SGPNE payoffs.

(b) Since (Abide, Infringe) and (Infringe, Abide) are Nash equilibria of the

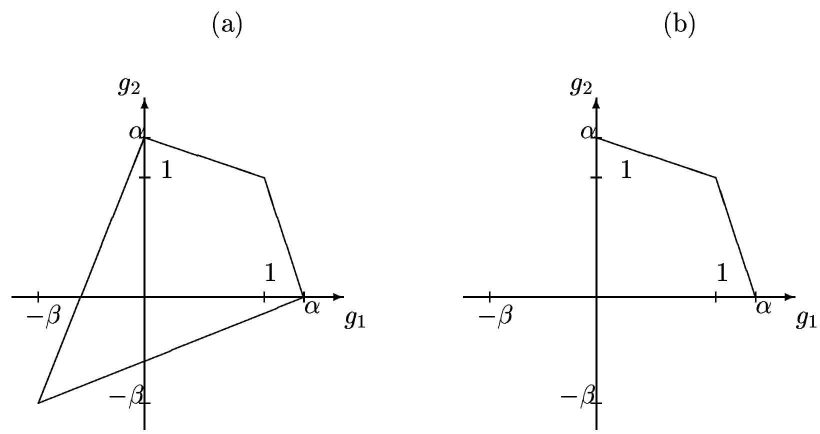

Figure 4: Payoffs for Question 4a

stage game, giving player 1 and 2, respectively, their minmax payoffs it is possible to support the outcome (Abide, Abide) in every period using Nash reversion strategies. Consider the following strategy for both players:

Play Abide on the first period. For later periods, play Abide if in all previous periods the play has been (Abide,Abide) or (Infringe, Infringe). After any other history, play abide if you were the first first to play Infringe; play infringe if the other player was the first to play infringe.

Given these strategies the equilibrium path will always be (Abide, Abide), and after a deviation by one player, the game will revert to the Nash equilibrium giving her the minmax payoff. It is possible to support these strategies as a SPGNE for large enough δ.

However, note that other punishment strategies are possible and these may require smaller δ's depending on the relative values of α and β. To write down a pair of strategies to do this, it will be useful to use the concept of "phases of play". There are two phases of play: Punishment phase and the normal phase. Play starts at normal phase. The transitions between two phases is determined as follows: irrespective of what the current phase is, if players conform to their prescribed actions, next phase is normal phase; irrespective of what the current phase is, if at least one player does not conform to her prescribed action, next phase is punishment phase. The structure of strategies that use this kind of a construct will be such that if someone deviates, there will be a one period punishment. If the defector does not let the other player punish her or if the other player refuses to punish, there will be another period of punishment, else, the game goes back to the "normal" play.

Now, If we specify an action for each player for each phase, we will have specified a complete strategy pair. Consider the following strategy for both players: *Play Abide in a normal phase and play Infringe in a punishment phase.* If β is large enough (i.e. punishment is severe enough), this strategies can be supported in a SGPNE with a δ that is smaller than that which is required to support the Nash reversion strategies described above.

Finally, note that the punishment phase does not have to end in one period. It is possible to force players to play punishment for longer periods. This will make it easier (require a smaller δ) to make them play their equilibrium strategies in a normal phase, but make it harder to make them conform to the punishment phase strategies. It might be the case that the smallest δ required is achieved by strategies of this sort. Also, it is possible to have more complicated punishment strategies. I will restrict attention to the two equilibrium strategy pairs described above.

(c) First consider the Nash reversion strategies. Given the other palyer's strategy, playing Abide at a subgame that starts after a history of no deviations will give the player a payoff of $1 + \delta \times 1 + \delta^2 \times 1 + ... = \frac{1}{1-\delta}$. Deviating will give a payoff of α for 1 period and then 0 forever after. Hence, a player will not want to deviate if $\alpha \leq \frac{1}{1-\delta}$, i.e. if $\delta \geq \frac{alpha-1}{\alpha}$. We do not have to check anything else, because the punishment strategies are Nash equilibria of the stage game.

To support the second proposed strategy pair, we need only to make sure each player plays their prescribed action in each phase. Also remember that we need to check only one period deviations, i.e. we will assume that after each deviation there will be one punishment phase an then the play returns to (Abide, Abide). Also, since the game is symmetric, it is enough to do the checking for one player only. Playing Abide in a normal phase leads to a payoff of 1 every period. Deviating in a normal phase gives one period of α, one period of $-\beta$ and then 1 forever after. Therefore, the player is willing to conform in a normal phase if and only if $\frac{1}{1-\delta} \geq \alpha + \delta \times (-\beta) + \frac{\delta^2}{1-\delta}$, ie, $\delta \geq \frac{\alpha-1}{1+\beta}$. Finally, if the player plays Infringe in a punishment phase, he will get $-\beta$ for one period and then 1 forever after and if she deviates and plays Abide, she will get one period of 0, one period of $-\beta$ and then 1 forever after. Therefore, she will conform to her punishment strategy only if $-\beta + \frac{\delta}{1-\delta} \geq 0 + \delta \times (-\beta) \times \frac{\delta^2}{1-\delta}$, i.e. if $\delta \geq \frac{\beta}{1+\beta}$. Hence, the smallest δ required to support this equilibrium is $\frac{\alpha-1}{1+\beta}$ whenever $\alpha - 1 \geq \beta$ and is $\frac{\beta}{1+\beta}$, otherwise. Note that in fact, in both cases the smallest delta required to support this equilibrium is higher than that which is required to support the Nash reversion equilibrium described above. However, since these bounds for δ involve more interesting dynamics, i will focus on them for the comparative statics analysis.

If it is the case that $\alpha - 1 \geq \beta$, then, a further increase in α will make the minimum feasible δ bigger, and hence will make it harder to support the equilibrium. This is because α is the reward for deviation and bigger α makes deviations more desirable. If β increases, but we are in the range where $\alpha - 1 \leq \beta$, minimum feasible δ decreases. That is because $-\beta$ is the punishment payoff, and the severer the punishment the less patience it takes to keep players on the equilibrium path. Finally, if β is in the range where $\alpha - 1 \geq \beta$, then minimum feasible δ increase with an increase in β. The reason is in a punishment phase punisher also gets a payoff of $-\beta$ and as this number gets smaller it gets harder to make the punishers conform to their punishment phase strategies. Finally, note that a higher minimum feasible δ means that cooperation is less likely.

Question 5

(a)Let S_K and S_C stand for the strategy spaces and u_K and u_C stand for the utility functions of the king and the citizen, respectively. Then, $S_K = S_C = [0,1]$. Also, for all $x \in S_C$ and $t \in S_K$, $u_C(x,t) = 1 + x(1-2t)$ and $u_K(x,t) = 2xt$.

To find the Nash equilibria of the game first note that the best response of the citizen to any tax rate above $\frac{1}{2}$ is to invest $x = 0$, to any tax rate below $\frac{1}{2}$ is to invest $x = 1$, and she is indifferent among all levels of investment whenever $t = \frac{1}{2}$. On the other hand, the king's unique strict best response to any $x > 0$ is to set $t = 1$. From this we can conclude that no Nash equilibrium can involve any positive investment by the citizen. Because if (x,t) is an equilibrium with $x > 0$, then it must be the case that $t = 1$ - otherwise king has an incentive to deviate. But when $t = 1$, the citizen has an incentive to deviate to $x = 0$. Therefore, in any Nash equilibrium $x = 0$. Then, the king is indifferent among all tax rates. But to ensure that the citizen does not have an incentive to deviate to $x = 1$ we need $t \geq \frac{1}{2}$. Hence (x,t)=(0,t) is a Nash equilibrium of this game for every $t \geq \frac{1}{2}$, and there are no other Nash equilibria.

This is a commitment problem. Since the king cannot commit not to confiscate all the production, the citizen does not produce anything.

(b) Above, we already specified the best responses of the citizen to every tax rate t. Given this, the king will get 0 payoff if he sets $t > \frac{1}{2}$, but it is possible for him to achieve positive payoff with smaller tax rates. Hence, in equilibrium we must have $t \leq \frac{1}{2}$. Moreover, no $t < \frac{1}{2}$ can be supported in equilibrium. To see this note that whenever $t < \frac{1}{2}$, citizen's unique best response is to set $x = 1$, and it is always possible to increase t a tiny bit while still keeping it strictly less than $\frac{1}{2}$. The citizen's best response to this new tax rate will still be $x = 1$, hence the king will get a strictly higher payoff by deviating to this slightly higher tax rate. These arguments establish that the unique SGPNE of this game involves $t = \frac{1}{2}$.

As argued above, when $t = \frac{1}{2}$, any level of x is a best response for the citizen. However, no $x < 1$ can be supported in equilibrium. To see why, note that the king is getting $2x\frac{1}{2} = x$ in this equilibrium. On the other hand any $t' < \frac{1}{2}$ will induce player 1 to play her unique best response $x = 1$. Then the king will get $2t'$. Whenever $x < 1$, it is possible to find t' close enough to but still strictly less than $\frac{1}{2}$ such that $2t' > x$. Therefore, the king will have an incentive to deviate to this tax rate t'. (Observe that these arguments would not go through if we did not ask for subgame perfection. Since the tax rate t' is off the equilibrium path, we could not require the citizen to react optimally to that and an equilibrium path in which $t = \frac{1}{2}$ and $x < 1$ could be supported with an incredible threat from the citizen to pick, say, $x = \frac{3}{4}$ no matter what t is, as long as it is less that

$\frac{1}{2}$).

Therefore the unique subgame perfect Nash equilibrium of this game is as follows: $t = \frac{1}{2}$; $x = 1$ if $t \leq \frac{1}{2}$ and $x = 0$ otherwise. The ability to commit on the part of the king makes positive production possible.

(c)The outcome will be socially efficient whenever $x = 1$, because this maximizes the total payoff to the citizen and the king. Assume we are trying to support an equilibrium in which $(1, \tilde{t})$ is played every period. Note that we cannot have $\tilde{t} > \frac{1}{2}$ because that would give the citizen a total discounted payoff less than her minmax payoff which is 1. Consider the following strategies:

Citizen: Play $x = 1$ in the first period, then keep playing $x = 1$ if the tax rate has been $\tilde{t}$ for all previous periods, otherwise revert to playing $x = 0$ forever (i.e. revert to her strategy in the Nash equilibrium (0,1) of the stage game.

King: Set $t = \tilde{t}$ in the first period. Then keep setting $t = \tilde{t}$ if the investment by the citizen has been $x = 1$ in all previous periods, otherwise revert to $t = 1$ forever (ie, play his part in the NE of the stage game). (Remark: Reverting to the (x,t)=(0,1) equilibrium of the stage game is arbitrary. Any other Nash equilibrium of the stage game would do the trick in this particular example, because they all give the same payoff to both players).

First note that the citizen will never have an incentive to deviate because given that $\tilde{t} \leq \frac{1}{2}$ there is no deviation that would make her strictly better off even in the one period stage game. The king is getting $2\tilde{t} + 2\tilde{t} \times \delta + ... = \frac{2\tilde{t}}{1-\delta}$ in equilibrium. His best deviation would be to confiscate everything, which would give him a payoff of 2 for one period and then 0 forever after. Therefore, he will not want to deviate as long as $\frac{2\tilde{t}}{1-\delta} \geq 2$, or equivalently $\delta \geq 1 - \tilde{t}$.

(d)The payoff of the citizen is decreasing in t, therefore, she will be best of when t is set to its smallest supportable value. By the previous part, it must be the case that $\delta \geq 1 - \tilde{t}$. Therefore, for a given δ, the smallest tax rate that can be supported is $1 - \delta$, and that will give the citizen a payoff of $2(1 - t) = 2(1 - 1 + \delta) = 2\delta$.

(e)Locke had argued that if people were really as opportunistic and self-interested as Hobbes made out, then who would be such a fool as to want to live under a government in which a single person was all-powerful? This would just be a recipe for exploitation, and one might do better fending for oneself with no such government. Hobbes could (and to an extent did) reply that the sovereign would have a self-interested reason not to exploit the citizens too much, as doing so would reduce their output and thus his tax revenues. The repeated game here shows how in principle the sovereign credibly could commit to a nonconfiscatory tax rate by "bonding" her reputation.